Homework 6, Problem 2

Consider the function

$$f(z) = \left(\frac{z}{1-z}\right)^{1/3} \frac{1}{1+z^2}.$$

In general, if α is not an integer, then z^α has branch points at 0 and ∞. Thus, $f(z)$ has branch points at the values z such that

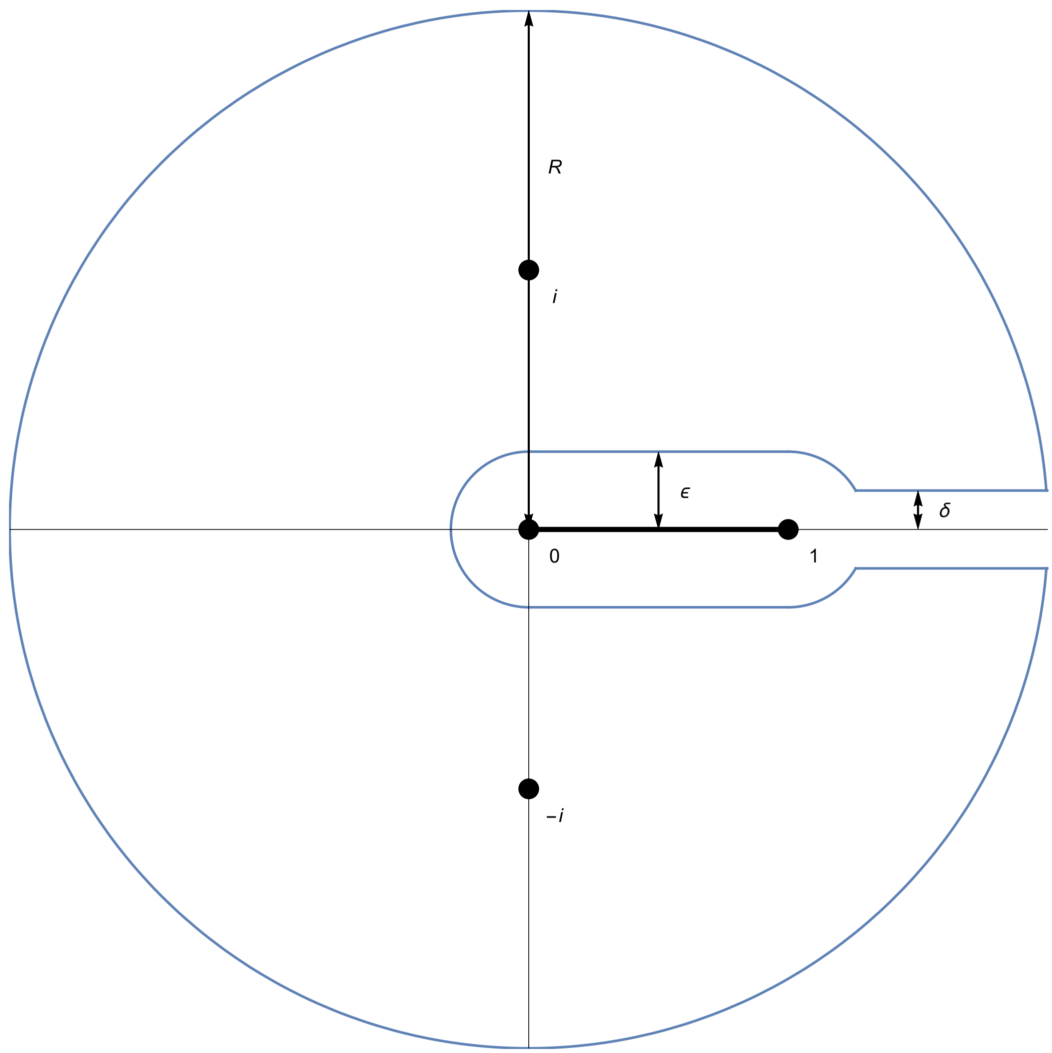
$$\frac{z}{1-z} = 0 \text{ or } \infty.$$

These are $z = 0$ and $z = 1$. Therefore, if we make the horizontal branch cut from 0 to 1, $f(z)$ is a well-defined single-valued analytic function away from this cut.

We wish to compute

$$I = \int_0^1 f(x)dx.$$

For this, start with the contour Γ



with the intention of taking ϵ and δ both to 0 and R to infinity. The integral over this contour does not change if we take δ as small as we want because $f(z)$ is analytic in the region to the right of 1. Thus, we can assume $\delta = 0$, so the integrals over the two segments δ away from the horizontal axis cancel out. Now, by the residue theorem,

$$\begin{aligned}2\pi i(\mathrm{Res}(i)+\mathrm{Res}(-i)) &= \int_\Gamma f(z)dz \\ &= \int_{\epsilon i}^{1+\epsilon i} f(x)dx + \int_{\pi/2}^{-\pi/2} f(1+\epsilon e^{i\theta})i\epsilon e^{i\theta}d\theta + \int_{1-\epsilon i}^{-\epsilon i} f(x)dx + \int_{3\pi/2}^{\pi/2} f(\epsilon e^{i\theta})i\epsilon e^{i\theta}d\theta \\ &\quad + \int_0^{2\pi} f(Re^{i\theta})iRe^{i\theta}d\theta\end{aligned}$$

We calculate each of these terms separately.

(i) As ϵ goes to 0, the first term,

$$\int_{\epsilon i}^{1+\epsilon i} f(x)dx,$$

goes to I.

(ii) The next term is

$$\begin{aligned}\int_{\pi/2}^{-\pi/2} f(1+\epsilon e^{i\theta})i\epsilon e^{i\theta}d\theta &= \int_{\pi/2}^{-\pi/2}\left(\frac{1+\epsilon e^{i\theta}}{-\epsilon e^{i\theta}}\right)^{1/3}\frac{1}{1+(1+\epsilon e^{i\theta})^2}i\epsilon e^{i\theta}d\theta \\ &= \epsilon^{2/3}\int_{\pi/2}^{-\pi/2}\left(-e^{-i\theta}-\epsilon\right)^{1/3}\frac{1}{1+(1+\epsilon e^{i\theta})^2}ie^{i\theta}d\theta.\end{aligned}$$

This integral therefore goes to 0 as ϵ goes to 0.

(iii) The next term

$$\int_{1-\epsilon i}^{-\epsilon i} f(x)dx$$

does not simply go to $-I$ as ϵ goes to 0. Traveling along the semicircle from $1+i\epsilon$ to $1-i\epsilon$, we pick up a factor of $e^{2\pi i/3}$. To see this, write $z=t-i\epsilon$ and

$$w=\frac{z}{1-z}=-1+\frac{1}{1-z}=-1+\frac{1}{1-t+i\epsilon}=-1+\frac{1-t-i\epsilon}{(1-t)^2+\epsilon^2}=\left(-1+\frac{1-t}{(1-t)^2+\epsilon^2}\right)-i\left(\frac{\epsilon}{(1-t)^2+\epsilon^2}\right).$$

As ϵ goes to zero, w approaches

$$-1+\frac{1-t}{(1-t)^2}=-1+\frac{1}{1-t}=\frac{t}{1-t}.$$

However, it approaches from below the horizontal axis, i.e., from the 4th quadrant. If we write in polar coordinates $w=re^{i\theta}$, then the point is that, by continuity, θ approaches 2π as ϵ approaches 0. Therefore, $w^{1/3}$ approaches

$$(\lim_{\epsilon\to 0} r)^{1/3}e^{i(\lim_{\epsilon\to 0}\theta)/3}=\left(\frac{t}{1-t}\right)^{1/3}e^{2\pi i/3}.$$

Therefore,

$$\int_{1-\epsilon i}^{-\epsilon i} f(x)dx$$

approaches $-e^{2\pi i/3}I$.

(iv) The next term is

$$\begin{aligned}\int_{3\pi/2}^{\pi/2} f(\epsilon e^{i\theta})i\epsilon e^{i\theta}d\theta &= \int_{3\pi/2}^{\pi/2}\left(\frac{\epsilon e^{i\theta}}{1-\epsilon e^{i\theta}}\right)^{1/3}\frac{1}{1+(\epsilon e^{i\theta})^2}i\epsilon e^{i\theta}d\theta \\ &= \epsilon^{4/3}\int_{3\pi/2}^{\pi/2}\left(\frac{e^{i\theta}}{1-\epsilon e^{i\theta}}\right)^{1/3}\frac{1}{1+(\epsilon e^{i\theta})^2}ie^{i\theta}d\theta.\end{aligned}$$

This integral therefore goes to 0 as ϵ goes to 0.

(v) The last term is

$$\begin{aligned}\int_0^{2\pi} f(Re^{i\theta})iRe^{i\theta}d\theta &= \int_0^{2\pi}\left(\frac{Re^{i\theta}}{1-Re^{i\theta}}\right)^{1/3}\frac{1}{1+(Re^{i\theta})^2}iRe^{i\theta}d\theta \\ &= \int_0^{2\pi}\left(\frac{e^{i\theta}}{1-Re^{i\theta}}\right)^{1/3}\frac{R^{4/3}}{1+(Re^{i\theta})^2}ie^{i\theta}d\theta.\end{aligned}$$

The size of the integrand grows on the order of

$$\frac{1}{R}\frac{R^{4/3}}{R^2} = \frac{1}{R^{5/3}}$$

as R grows. Taking R to infinity, the integral above therefore goes to 0.

Putting this all together, we have

$$\begin{aligned}
I(1-e^{2\pi i/3}) = I - e^{2\pi i/3} I &= 2\pi i(\operatorname{Res}(i) + \operatorname{Res}(-i)) \\
&= 2\pi i \left(\lim_{z\to i} f(z)(z-i) + \lim_{z\to -i} f(z)(z+i) \right) \\
&= 2\pi i \left(\left(\frac{i}{1-i}\right)^{1/3} \frac{1}{i+i} + \left(\frac{-i}{1+i}\right)^{1/3} \frac{1}{-i-i} \right) \\
&= 2\pi i \left(\left(\frac{-1+i}{2}\right)^{1/3} \frac{1}{2i} - \left(\frac{-1-i}{2}\right)^{1/3} \frac{1}{2i} \right) \\
&= 2\pi i \left(\left(\frac{1}{\sqrt{2}} e^{3\pi i/4}\right)^{1/3} \frac{1}{2i} - \left(\frac{1}{\sqrt{2}} e^{5\pi i/4}\right)^{1/3} \frac{1}{2i} \right) \\
&= \frac{\pi}{2^{1/6}} \left(e^{\pi i/4} - e^{5\pi i/12} \right).
\end{aligned}$$

Solving for I,

$$\begin{aligned}
I &= \frac{\pi}{2^{1/6}} \frac{e^{\pi i/4} - e^{5\pi i/12}}{1 - e^{2\pi i/3}} \\
&= \frac{\pi}{2^{1/6}} \frac{e^{\pi i/3}(e^{-\pi i/12} - e^{\pi i/12})}{e^{\pi i/3}(e^{-\pi i/3} - e^{\pi i/3})} \\
&= \frac{\pi}{2^{1/6}} \frac{\sin(\pi/12)}{\sin(\pi/3)}.
\end{aligned}$$

Using special angle formulas, we also have

$$I = \frac{2^{1/3}\pi}{3+\sqrt{3}} = \frac{(3-\sqrt{3})\pi}{3\cdot 2^{2/3}}.$$